Q. Let X denote the set of all possible sequences $x = \{x_n\}$. Define

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1+|x_n - y_n|} \text{ where } x,y \in X$$

Show that 'd' is metric on X

Soln We see that the series given in RHS is absolutely convergent since each term is $\leq$ the corresponding term of Geometric Series $\sum \frac{1}{2^n}$ which is convergent. Hence $d(x,y)$ is well defined.

[M1]: $\frac{1}{2^n} \frac{|x_n - y_n|}{1+|x_n - y_n|} \geq 0$

We know that $|x_n - y_n| \geq 0$

Also $1+|x_n - y_n| > 0$

so obviously $\frac{1}{2^n} \frac{|x_n - y_n|}{1+|x_n - y_n|} \geq 0$

$\Rightarrow d(x,y) \geq 0$.

[M2]: $|x_n - y_n| = 0$ iff $x_n = y_n$

$\Rightarrow \frac{1}{2^n} \frac{|x_n - y_n|}{1+|x_n - y_n|} = 0$ iff $x_n = y_n$

$\Rightarrow d(x,y) = 0$ iff $x_n = y_n$.

[M3]: $|x_n - y_n| = |y_n - x_n|$

$\Rightarrow \frac{1}{2^n} \frac{|x_n - y_n|}{1+|x_n - y_n|} = \frac{1}{2^n} \frac{|y_n - x_n|}{1+|y_n - x_n|}$

$\Rightarrow d(x,y) = d(y,x)$

Similarly (M4) $d(x,y) \leq d(x,z) + d(z,y)$

$\{$ use inequality $0 < \alpha < \beta$

$\Rightarrow \alpha(1+\beta) \leq \beta(1+\alpha) \Rightarrow \frac{\alpha}{1+\alpha} \leq \frac{\beta}{1+\beta}$?

$\alpha = |x_n - z_n + z_n - y_n|$
$\beta = |x_n - z_n| + |z_n - y_n|$

Hence (X, d) is a metric space.

<u>Note</u>: The set of all sequences of all real nos. with the above metric $\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1+|x_n - y_n|}$ is called FRECHET SPACE and is denoted by R_ω.

Bounded Metric Space

<u>Defn.</u> Let (X, d) be a metric space. Then 'X' is said to be bounded if there exists a positive number M such that $d(x,y) \leq M$ for every pair of points x and y of 'X';

All metric spaces which are not bounded are called unbounded.

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1+|x_n - y_n|} \leq 2.$$

Theorem - In a Metric space (X,d) finite union of bounded sets are also bounded.

Proof: Let $A_1, A_2, \dots, A_n$ be bounded sets in (X,d).
~~There~~ there exists $a_1, a_2, \dots, a_n \in X$ and $k_1, k_2, \dots, k_n > 0$ such that $d(x, a_m) < k_m \; \forall \; x \in A_m \; (m = 1, 2, 3, \dots, n)$

Let $A = \bigcup_{m=1}^{n} A_m$. then we have

to prove that A is bounded.
For this let $x \in A$, then $x \in A_m$ for some m $(1 \leq m \leq n)$

Now $d(x, a_1) \leq d(x, a_m) + d(a_m, a_1)$

$$\leq k_m + d(a_m, a_1)$$

$$< \max_{1 \leq m \leq n} k_m + \max_{1 \leq m \leq n} d(a_m, a_1)$$

$$\doteq k$$

$\Rightarrow d(x, a_1) < k$
Hence A is a bounded set

Proved

Diameter of a non-empty Set
The diameter of a non-empty subset $A \subseteq X$ denoted by $\delta(A)$ is defined by

$$\delta(A) = \sup.[d(a,b) : a, b \in A]$$

If $\delta(A) < \infty$, then diameter is said to be finite otherwise infinite.